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Introduction to Differential Equations

A differential equation is an equation that contains one or more derivative of a function. This handout will serve as an introduction to differential equations and will cover topics including identifying differential equations, solving first-order equations, verifying solutions to differential equations, and checking to determine if a solution is lost. This handout will specifically focus on solving first-order linear and separable equations.

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Order of a Differential Equation

The order of a differential equation refers to an equation's highest derivative. For example, if the highest derivative is f'', then the equation is a second order equation; similarly, if the highest derivative is $f^{(4)}$, then the equation is a fourth order equation. Note that $f^{(3)}$ is not the same as f^3 because the parenthesis around the 3 in the exponent denotes that it is the third derivative, but the lack of parenthesis denotes that it is raised to the third power.

Types of Differential Equations

It is important to be able identify several types of differential equations. The order and type of a differential equation will determine how that equation is solved. Often, algebra must be used to rearrange the differential equation into a form that is easy to identify.

<u>Linear</u>

A differential equation is linear if it can be arranged in the following form:

$$u_n y^{(n)} + u_{n-1} y^{n-1} + \cdots + u_1 y' + u_0 y = f(x)$$

$f(x)$, also known as the forcing function, is a function that contains only x's and is usually placed on the opposite side of the equal sign from the terms that involve y. Note that u can either be a function of x or a constant, but it does not contain y. In a linear differential equation, all derivatives of a function, y, can be multiplied by u, added together, and equal to the forcing function. In a linear differential equation, y and its derivatives cannot be inside functions of y, such as, $\cos(y')$ or $\frac{1}{y}$.

Example 1: $x^2 y^{(3)} + \sinh(3x)\, y'' + y' + 4y = e^{2x}$

This differential equation is linear because y and its derivatives are added together on one side of the equation and equal to a function of x on the other side.

Example 2: $\sin(x)\, y'' + 3\cos(y') - \frac{1}{y} = x^2$

Note that this differential equation is not linear because y' is inside of a cosine function, and y is in the bottom of a fraction.

Separable

A differential equation is separable if it can be arranged in the following form:

$$f(x)dx = g(y)dy.$$

A differential equation can be separable only if the equation is first-order. In a separable differential equation, all functions of x are multiplied only by a dx, and all functions of y are multiplied only by a dy. y' should be rewritten as $\frac{dy}{dx}$ to make this possible.

Example:

$$y' = \frac{\cos(x)}{\sin(y)}$$

$$\frac{dy}{dx} = \frac{\cos(x)}{\sin(y)}$$

$$\sin(y)\, dy = \cos(x)\, dx$$

This differential equation is separable because both sides of the equation can be multiplied by $\sin(\mathrm{y})\mathrm{dx}$ to rearrange the equation as $\sin(\mathrm{y})\,\mathrm{dy} = \cos(\mathrm{x})\,\mathrm{dx}$, with all functions of y multiplied by only a dy, and all functions of x multiplied by only a dx.

Homogeneous Linear

A linear differential equation is homogeneous if the forcing function is equal to zero. Homogeneous linear differential equations can be written in the following form:

$$u_n \mathrm{y}^{(\mathrm{n})} + \mathrm{u}_{\mathrm{n}-1}\mathrm{y}^{(\mathrm{n}-1)} + \cdots + \mathrm{u}_1\mathrm{y}' + \mathrm{u}_0\mathrm{y} = 0.$$

Example: $\mathrm{y}' + 5\mathrm{y}'' - 6\mathrm{y} = 0$

This linear differential equation is homogeneous because its forcing function is equal to zero.

Identifying Differential Equations:

Identify the following equations as linear, separable, or homogeneous. Note: A differential equation can be more than one type, or no type at all.

1. $\mathrm{y}' - \sin(\mathrm{y})(\mathrm{x}^2 - 9) = 0$

   This equation is separable because it can be rewritten as $\frac{1}{\sin(\mathrm{y})}\mathrm{dy} = (\mathrm{x}^2 - 9)\mathrm{dx}$ with all functions of y multiplied by only dy and all functions of x multiplied by only dx. This equation is not linear because y is inside of a sine function.

2. $\sin(\mathrm{x})\,\mathrm{y}'' = -3\mathrm{y}(\mathrm{x}^2 - 4)$

   This equation is linear and homogeneous because it can be rewritten as $\sin(\mathrm{x})\,\mathrm{y}'' + 3(\mathrm{x}^2 - 4)\mathrm{y} = 0$ with the forcing function equal to zero. Note, there is no y' term because it is multiplied by zero. This equation is not separable because it is not first-order.

3. $\mathrm{y}'' + \cos(\mathrm{x})\,\mathrm{e}^{\mathrm{y}} = \mathrm{x}^2 - 4$

   This equation is not linear because the function e^{y} cannot be expressed linearly. Additionally, the equation is not separable because it is not first-order. This function is also not homogeneous because it is both non-linear and has the forcing function $x^2 - 4$.

4. $y' + \ln(x)\,y = -y$

This equation is linear and homogeneous because it can be written as $y' + (1 + \ln(x))y = 0$. It is also separable because it can be written as $\frac{1}{y}dy = -(1 + \ln(x))\,dx$.

Solving Separable Differential Equations

Step 1: If necessary, replace y' with $\frac{dy}{dx}$.

Step 2: Move all xs on one side of the equation and all ys on the other side.

Step 3: Integrate both sides of the equation: $\int g(y)dy = \int f(x)dx$.

When integrating, both constants of integration can be combined and placed on one side of the equation; this is generally the side that contains x.

Step 4: Solve for y.

Step 5: If initial conditions are given, use to solve for the constant(s).

Example: Solve the following differential equation using separation:

$$y' = \frac{\cos(x)}{y^2} \quad \text{where } y(0) = 1$$

Step 1: If necessary, replace y' with $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\cos(x)}{y^2}$$

Step 2: Move all xs on one side and all ys on the other side:

$$y^2 dy = \cos(x)\,dx$$

Step 3: Integrate both sides of the equation:

$$\int y^2 dy = \int \cos(x)\,dx$$

$$\frac{1}{3}y^3 = \sin(x) + C$$

Step 4: Solve for y:

$$y = \sqrt[3]{3\sin(x) + 3C}$$

When C is multiplied by 3, it is still a constant, so it can be represented by a new constant, K.

$$3C = K$$

$$y = \sqrt[3]{3\sin(x) + K}$$

Step 5: If initial conditions are given, use to solve for the constant(s):

$y(0) = 1 = \sqrt[3]{3\sin(0) + K}$, therefore $1 = \sqrt[3]{K}$ and $K = 1$

The solution to this equation is $y = \sqrt[3]{3\sin(x) + 1}$

Solving First Order Linear Differential Equations

Step 1: Rearrange the differential equation into linear form and make the function multiplied onto the y' term 1: $y' + p(x)y = f(x)$.

Step 2: Representing the coefficient of the y term as $p(x)$, find the integrating factor, μ, for the differential equation by using the formula $\mu = e^{\int p(x)dx}$.

Step 3: Multiply both sides of the equation by μ to rewrite the equation in the form: $\mu y' + \mu p(x)y = \mu f(x)$.

Step 4: Because the previous step resembles the product rule of differentiation, rewrite the equation in the form $\frac{d}{dx}(\mu y) = \mu f(x)$.

Step 5: Move dx to the other side of the equation and solve the separable differential equation.

Step 6: If initial conditions are given, use to solve for the constant(s).

Example: Solve the following linear differential equation:

$$y' = -\left(\frac{1}{x+1}\right)y + x^2 + x + 5 \qquad \text{where } y(0) = 1$$

Step 1: Rearrange the differential equation in linear form and make the function multiplied onto the y' term 1:

$$y' + \left(\frac{1}{x+1}\right)y = x^2 + x + 5$$

Step 2: Representing the coefficient of the y term as $\mathrm{p(x)}$, find the integrating factor, μ, for the differential equation:

$$\mathrm{p(x)} = \frac{1}{\mathrm{x+1}}$$

$$\mu = e^{\int p(x)dx} = e^{\int \frac{1}{x+1}dx} = e^{\ln|x+1|} = x+1$$

Step 3: Multiply both sides of the differential equation by μ:

$$(x+1)y' + (x+1)\left(\frac{1}{x+1}\right)y = (x+1)(x^2+x+5)$$

$$(x+1)y' + y = (x+1)(x^2+x+5)$$

Step 4: Rewrite the equation in the form $\frac{d(\mu y)}{dx} = \mu f(x)$:

$$\frac{d((x+1)y)}{dx} = (x+1)(x^2+x+5)$$

Step 5: Move dx to the other side of the equation and solve the separable differential equation:

$$d\big((x+1)y\big) = (x+1)(x^2+x+5)dx$$

$$\int d((x+1)y) = \int (x+1)(x^2+x+5)dx$$

$$\int d((x+1)y) = \int (x^3+2x^2+6x+5)dx$$

$$(x+1)y = \frac{1}{4}x^4 + \frac{2}{3}x^3 + 3x^2 + 5x + C$$

$$y = \frac{\frac{1}{4}x^4 + \frac{2}{3}x^3 + 3x^2 + 5x + C}{x+1}$$

Step 6: If initial conditions are given, use to solve for the constant(s):

$$y(0) = 1 = \frac{\frac{1}{4}(0)^4 + \frac{2}{3}(0)^3 + 3(0)^2 + 5(0) + C}{(0)+1} = C$$

$$C = 1$$

$$y = \frac{\frac{1}{4}x^4 + \frac{2}{3}x^3 + 3x^2 + 5x + 1}{x+1}$$

Verifying Solutions to Differential Equations

A solution to a differential equation is valid if it can be substituted into each side of the original equation and simplified to produce equal values on both sides of the equation.

Step 1: Note the order of the differential equation to determine how many times to take the derivative of the solution.

Step 2: Substitute the derivatives into the original differential equation.

Step 3: Simplify the expression to check if both sides of the equal sign are the same.

Example: Verify that $y = e^{8x} + e^{2x}$ is a solution to the differential equation $y'' - 10y' + 16y = 0$.

Step 1: Note the order of the differential equation to determine how many times to take the derivative of the solution. In this case, the equation is second order, so find the first and second derivatives of the solution:

$$y' = 8e^{8x} + 2e^{2x}$$
$$y'' = 64e^{8x} + 4e^{2x}$$

Step 2: Substitute the derivatives into the differential equation:

$$(64e^{8x} + 4e^{2x}) - 10(8e^{8x} + 2e^{2x}) + 16(e^{8x} + e^{2x}) = 0$$

Step 3: Simplify the expression to check if both sides of the equal sign are the same:

$$64e^{8x} + 4e^{2x} - 80e^{8x} - 20e^{2x} + 16e^{8x} + 16e^{2x} = 0$$
$$(64 - 80 + 16)e^{8x} + (4 - 20 + 16)e^{2x} = 0$$
$$0e^{8x} + 0e^{2x} = 0$$

In this case, $0 = 0$; therefore, the statement is true, and the given function is a solution to the given differential equation.

Finding Lost Solutions to Differential Equations

When solving a separable differential equation, it is possible that a solution may be lost in the process. When dividing both sides of the equation by an expression involving a variable, it is possible that there is a value for the variable that makes the denominator of the fraction equal to zero. This value may be a lost solution.

Step 1: Find potential lost solution(s).

Step 2: Substitute y in the known solution with the potential lost solutions.

Step 3: Solve for x to see if the known solution can produce the potential lost solution.

Example: The differential equation $y' = y^2 - 4$ has a solution $y = 2\frac{1+ce^{4x}}{1-ce^{4x}}$.

Find any lost solutions.

This differential equation is separable and can be rewritten as $\frac{dy}{dx} = y^2 - 4$, so in order to get all functions of y separate from functions of x, $y^2 - 4$ must be divided on both sides of the equation.

$$\frac{dy}{y^2 - 4} = dx$$

Step 1: Find potential lost solution(s).

When dividing by $y^2 - 4$, the values for y, which make this expression equal to zero, need to be checked because there cannot be a zero in the denominator. In this example, both $y = 2$ and $y = -2$ make the expression equal to zero, so these quantities may be lost solutions. In order to test if a solution is lost, first verify if both values are in fact solutions of the differential equation $y' = y^2 - 4$.

$$y = 2, -2$$

$$y' = 0, 0$$

$$0 = (\pm 2)^2 - 4$$

$$0 = 0$$

Once both solutions are verified using this method, these solutions need to be tested as lost through substitution with the known solution.

Step 2: Substitute y in the known solution with the potential lost solutions:

$$y = 2\frac{1+ce^{4x}}{1-ce^{4x}}:$$

$$-2 = 2\frac{1+ce^{4x}}{1-ce^{4x}} \qquad 2 = 2\frac{1+ce^{4x}}{1-ce^{4x}}$$

Step 3: Solve for x to see if the known solution can produce the potential lost solution:

$$-1 = \frac{1+ce^{4x}}{1-ce^{4x}} \qquad\qquad 1 = \frac{1+ce^{4x}}{1-ce^{4x}}$$

$$ce^{4x} - 1 = 1 + ce^{4x} \qquad\qquad 1 - ce^{4x} = 1 + ce^{4x}$$

$$0 = 2 \qquad\qquad 0 = 2ce^{4x}$$

$y = -2$ is a lost solution because there is no x value of the known solution that satisfies $y = -2$. However, $y = 2$ is not lost because if c is equal to zero, then the solution can be satisfied by the known solution.