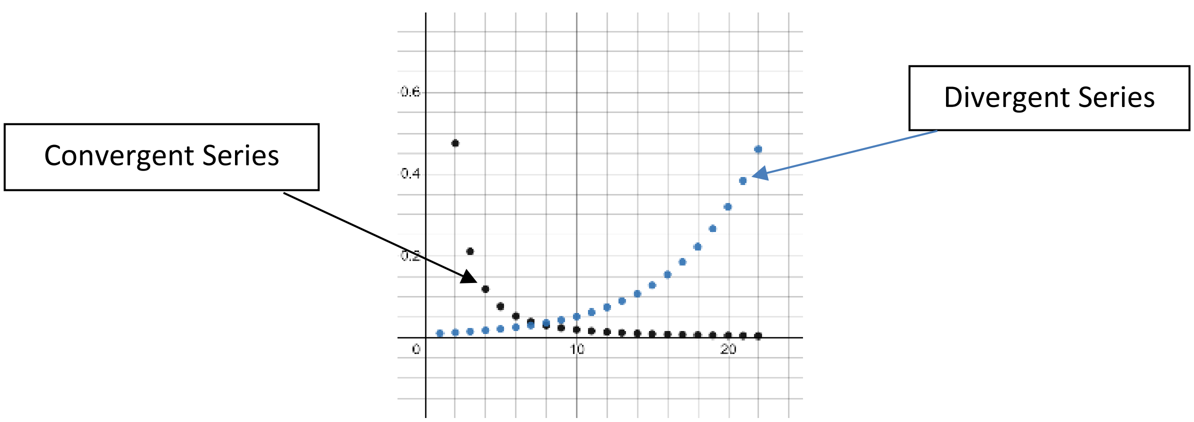
Converging and Diverging Series

A series is the summation of an infinite number of sequential terms. This concept plays an important role in mathematics and other quantitative areas of studies, such as statistics, finance, physics, etc., as it helps understand and predict behaviors using known patterns. This handout will cover seven tests that can be used to determine divergence or convergence, including:

- Geometric and P-Series Tests
- Direct Comparison Tests
- Divergence Tests
- Integral Tests
- Ratio Tests
- Root Tests

Convergent and Divergent Series

When working with series, it is important to define whether the series converges or diverges. When the limit of a series approaches a real number (i.e., the limit exists), it displays convergent behavior. As a result, an approximation can be evaluated for that given series. However, if the limit does not exist or is equal to infinity, that series displays divergent behavior. This is shown by the graph below appearing to approach towards infinity.



To find whether a series is either converging or diverging, a test can be conducted to determine the behavior of the given series. To start, the Geometric Series and P-series Tests are used when the given series looks identical to its respective test. If the given series does not apply to these two series, conduct the Direct Comparison Test. However, if the previous tests are inapplicable, try the Divergence Test. If it turns out to be inconclusive, attempt the Integral Test. If it looks difficult to integrate and/or contains a factorial, use the Ratio Test. Alternatively, if the series has an expression to the power of n, then use the Root Test.

Geometric Series and P-Series Test

The tests used to determine the behavior of a Geometric and P-series follow a specific equation format. A Geometric Series is the sum of a set of terms, where each term, a_n, is being multiplied by some ratio, r^n. The Geometric Series Test compares r with 1 to determine its behavior. A P-series is the sum of a set of terms, where the denominator of each term, $\frac{1}{n}$, is raised to some p value. Similarly, the P-series Test compares p with 1 to determine its behavior.

Geometric Series Test	P-Series Test
$$\sum_{n=1}^{\infty} a_1 (r)^n$$	$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
Diverges: For $\lvert r \rvert \geq 1$	Diverges: For $0 < p \leq 1$
Converges: For $\lvert r \rvert < 1$; Converges to $\frac{a_1}{1-r}$	Converges: For $p > 1$

Steps to apply:

Step 1: Determine the type of series given.
Step 2: Determine the value of r or p based on the type of series.
Step 3: Use the appropriate condition to determine its behavior.
Step 4: If it is a converging Geometric Series, use $\frac{a_1}{1-r}$ to find what it converges to.

Example A: Determine if the series converges or diverges. If it converges, determine where the series converges.

$$\sum_{n=1}^{\infty} 7\left(\frac{3}{8}\right)^{n-1}$$

Step 1: Determine the type of series given.

The series depicts a number to the power of some n variable. Therefore, this is considered a Geometric Series.

Step 2: Determine the value of r or p based on the type of series.

In this case, $\mathrm{r} = \frac{3}{8}$ and $\mathrm{a}_1 = 7$.

Step 3: Use the appropriate condition to determine its behavior.

Based on the condition, $|\mathrm{r}| < 1$, the given series must converge.

Step 4: If it is a converging Geometric Series, use $\frac{\mathrm{a}_1}{1-\mathrm{r}}$ to find what it converges to.

In this case, $\frac{\mathrm{a}_1}{1-\mathrm{r}} \rightarrow \frac{(7)}{1-\left(\frac{3}{8}\right)}$. As a result, the given series would converge to $\frac{56}{5}$.

__Example B:__ Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5}{\sqrt[2]{n^9}}$$

Step 1: Determine the type of series given.

The formula of a P-series is applicable if the numerator is 1.
Since the numerator is a constant, it can be factored out of the series:

$$5\sum_{n=1}^{\infty} \frac{1}{\sqrt[2]{n^9}}$$

Therefore, this series is a P-series.

Step 2: Determine the value of r or p based on the type of series.

The denominator, $\sqrt[2]{\mathrm{n}^9}$, can be rewritten as $\mathrm{n}^{\frac{9}{2}}$; therefore, $\mathrm{p} = \frac{9}{2}$.

Step 3: Use the appropriate condition to determine its behavior.

Based on the condition, $p > 1$, this series converges.

Using the P-series Test where, $p = \frac{9}{2}$, it can be determined that the given series converges.

Direct Comparison Test

If neither the P-series nor the Geometric Series Test is applicable to the given series, then another test is needed. The Direct Comparison Test compares the given series to a "known series", a series that is either a Geometric or P-series through modifying the given series. Since the modified series is derived from the given series, the behavior of both series should match. As a result, the behavior of the given series can be found through finding the behavior of the modified series.

Direct Comparison Test

When given an infinite series (a):

$$\sum_{n=1}^{\infty} a_n$$

Let the modified series (b) be:

$$\sum_{n=1}^{\infty} b_n$$

If $a_n \leq b_n$: When b_n converges, then a_n also converges.

If $a_n \geq b_n$: When b_n diverges, then a_n also diverges.

Steps to apply:

Step 1: Determine if the series can be compared to a P-series or Geometric Series by modifying the given series.

Step 2: Determine if the modified series is larger or smaller than the original series.

Step 3: Simplify if necessary. Find the value of r or p and determine its behavior. Refer to the previous section regarding Geometric and P-series if needed.

Example A: Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5^n}{9^n + 4}$$

Step 1: Determine if the series can be compared to a P-series or Geometric Series by modifying the given series.

Since there are terms to the power of the n variable, this is a good indication that it should be compared to a Geometric Series.

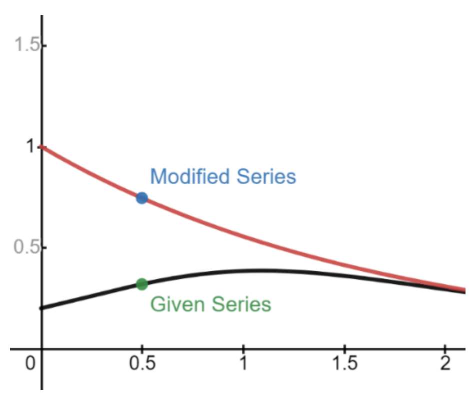
Step 2: Determine if the modified series is larger or smaller than the original series.

$$\sum_{n=1}^{\infty} \frac{5^n}{9^n} > \sum_{n=1}^{\infty} \frac{5^n}{9^n + 4}$$

The given series has an expression as its denominator. By removing the added term, the value of that expression would decrease. As a result, the value of the denominator would decrease, which would increase the value of the entire fraction.

The "$+ 4$" on the denominator does not drastically change the behavior of the series, and by removing that term, a larger series is produced. As a result, the modified series is $\sum_{n}^{\infty} \frac{5^n}{9^n}$.

The graph below shows that both series share the same behavior, as they both approach the same point. Showing that by finding the behavior of the modified series, the behavior of the original can also be found.



Step 3: Simplify if necessary. Find the value of r or p and determine its behavior.

The series, $\sum_{n=1}^{\infty} \frac{5^n}{9^n}$, can be rewritten as $\sum_{n=1}^{\infty} \left(\frac{5}{9}\right)^n$

From the modified series, $r = \frac{5}{9}$. Based on the conditions from the Geometric Series Test, if $r < 1$, the series must converge.

Since the larger series converges, via the Direct Comparison Test, the original series must also converge.

<u>**Example B:**</u> Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{\sqrt[3]{n^2} - 1}$$

Step 1: Determine if the series can be compared to a P-series or Geometric Series by modifying the given series.

The n variable is not placed as an exponent, unlike the previous example. Instead, n of different degrees are in the numerator and denominator, where the n with the highest degree is in the latter. As a result, this should be compared to a P-series.

Step 2: Determine if the modified series is larger or smaller than the original series.

Removing a subtracted term from the expression in the denominator increases the value of that expression, thus decreasing the value of the fraction.

$$\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{\sqrt[3]{n^2} - 1} > \sum_{n=1}^{\infty} \frac{4\sqrt{n}}{\sqrt[3]{n^2}}$$

Removing the "-1" from the denominator results in a smaller series. The fraction with the smaller denominator is always going to be bigger than the fraction with the larger denominator.

Step 3: Simplify if necessary. Find the value of r or p and determine its behavior.

$$\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{\sqrt[3]{n^2}} \to \sum_{n=1}^{\infty} 4\frac{(n)^{\frac{1}{2}}}{(n)^{\frac{2}{3}}}$$

Recall properties of exponents where $\mathrm{n}^{-\mathrm{a}} = \frac{1}{\mathrm{n}^{\mathrm{a}}}$ and $\mathrm{n}^{\mathrm{a}} * \mathrm{n}^{\mathrm{b}} = \mathrm{n}^{\mathrm{a}+\mathrm{b}}$

$$\sum_{n=1}^{\infty} 4\left(\frac{1}{n^{\frac{2}{3}} * n^{-\frac{1}{2}}}\right)$$

$$\sum_{n=1}^{\infty} 4\left(\frac{1}{n^{\frac{1}{6}}}\right)$$

From the modified series, $p = \frac{1}{6}$. Since the smaller series diverges, via the Direct Comparison Test, the original series must also diverge.

Divergence Test

If the given series cannot be compared to a Geometric or P-series, then the Divergence Test should be used. During this test, there will be times where L'Hopital's Rule (LHR) will be applied when the limit is $\frac{\infty}{\infty}$ or $\frac{0}{0}$.

Divergence Test Given $\sum_{n=1}^{\infty} a_n$: $$\lim_{n\to\infty} a_n$$ Diverges: If $\lim_{n\to\infty} a_n \neq 0$ Note: If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive. A different test should be used.

Steps to apply:

Step 1: Find the limit as n approaches infinity.
Step 2: Determine if the limit satisfies the test condition.

Example A: Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{7n^3 + 2n}{9 + 2n^2}$$

Step 1: Find the limit as *n* approaches infinity.

$$\sum_{n=1}^{\infty} \frac{7n^3 + 2n}{9 + 2n^2} \rightarrow \lim_{n \to \infty} \frac{7n^3 + 2n}{9 + 2n^2}$$

To find the limit, start by plugging in ∞ for n: $\lim\limits_{n \to \infty} \frac{7(\infty)^3+2(\infty)}{9+2(\infty)^2}$

For more information about computing limits, refer to ACE's Limit Handout.

Because the limit is $\frac{\infty}{\infty}$, LHR is applied.

$$\text{LHR: } \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

In this case, $\frac{f(n)}{g(n)} = \frac{7n^3+2n}{9+2n^2}$ and through LHR this results in: $\frac{f'(n)}{g'(n)} = \frac{21n^2+2}{4n}$

Plugging in ∞ for n: $\lim\limits_{n \to \infty} \frac{21(\infty)^2+2}{4(\infty)}$, this results in $\frac{\infty}{\infty}$ thus LHR is applied again.

In this case, $\frac{f(n)}{g(n)} = \frac{21n^2+2}{4n}$ and through LHR this results in: $\frac{f'(n)}{g'(n)} = \frac{42n}{4}$

Plugging in ∞ for n: $\lim\limits_{n \to \infty} \frac{42(\infty)}{4}$, this evaluates to ∞.

Step 2: Determine if the limit satisfies the test condition.

Since the $\lim\limits_{n \to \infty} \frac{7n^3+2n}{9+2n^2} \neq 0$, the series diverges.

Integral Test

If the Divergence Test proves to be inconclusive, then the Integral Test should be performed. As the name suggests, this test will require the use of integration.

Integral Test

Given a series:

$$\sum_{n=1}^{\infty} a_n$$

All n will be replaced with x.

If $\int_1^{\infty} f(x)\,dx$ converges, then the series converges.

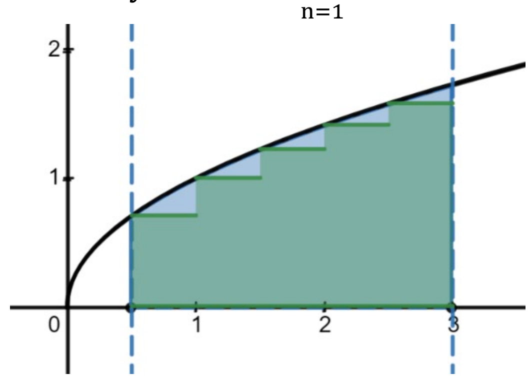
If $\int_1^{\infty} f(x)\,dx$ diverges, then the series diverges.

Converges: $-\infty < \int_1^{\infty} f(x)\,dx < \infty$

Diverges: $\int_1^{\infty} f(x)\,dx = \pm\infty$

Note: The number evaluated by the integral is not where the series converges to. [Refer to image to the right.]

$$\int f(x)\,dx > \sum_{n=1}^{\infty} a_n$$



Steps to apply:

Step 1: Replace all n with x.

Step 2: Integrate between 1 and ∞.

Step 3: Determine the series behavior based on test conditions.

<u>**Example A:**</u> Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^3}$$

Step 1: Replace all *n* with *x*.

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^3} \to \int_1^{\infty} \frac{x}{(x^2+1)^3}\,dx$$

Step 2: Integrate between 1 and ∞.

Integrate the definite integral. This will require the use of u-substitution.

For more information on integration, refer to ACE's Integral Handout.

$$u = (x^2 + 1), du = 2x\,dx$$

$$\frac{1}{2}\int_1^\infty \frac{1}{(x^2+1)^3}\,2x\,dx \rightarrow \frac{1}{2}\int_1^\infty \frac{1}{u^3}\,du \rightarrow \frac{1}{2}\int_1^\infty u^{-3}\,du$$

$$\frac{1}{2}\int_1^\infty u^{-3+1} \rightarrow \frac{1}{2}\int_1^\infty -\frac{1}{2} * u^{-2} \rightarrow -\frac{1}{4}(u^{-2})\Big|_1^\infty$$

$$-\frac{1}{4}\left([x^2+1]^{-2}\right)\Big|_1^\infty \rightarrow -\frac{1}{4}\left(\frac{1}{[x^2+1]^2}\right)\Big|_1^\infty$$

$$-\frac{1}{4}\left(\frac{1}{[(\infty)^2+1]^2} - \frac{1}{[(1)^2+1]^2}\right) \rightarrow -\frac{1}{4}\left(\frac{1}{\infty} - \frac{1}{2^2}\right) \rightarrow -\frac{1}{4}\left(0 - \frac{1}{4}\right) = \frac{1}{16}$$

Step 3: Determine the series behavior based on test conditions.

The integral evaluates to $\frac{1}{16}$ which is between $-\infty$ and ∞, thus the integral converges. Since the integral converges, the series must also converge.

Ratio Test

If the series looks difficult to integrate or contains a factorial, then the Ratio Test should be applied.

Ratio Test

Given $\sum_{n=1}^{\infty} a_n$

Let

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Converges: For $r < 1$

Diverges: For $r > 1$

Inconclusive: For $r = 1$

Steps to apply:

Step 1: Substitute n with $\mathrm{n}+1$ to find $\mathrm{a}_{\mathrm{n}+1}$ and divide by the original series, a_{n}.
Step 2: Simplify and solve the limit.
Step 3: Determine the behavior of the series based on the conditions listed in the test.

Example A: Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{(n+1)!}$$

Step 1: Substitute n with $\mathrm{n}+1$ to find $\mathrm{a}_{\mathrm{n}+1}$ and divide by the original series, a_{n}.

$$a_{n+1} = \sum_{n=1}^{\infty} \frac{2^{(n+1)}}{([n+1]+1)!} \rightarrow \sum_{n=1}^{\infty} \frac{2^{n+1}}{(n+2)!}$$

$$a_n = \sum_{n=1}^{\infty} \frac{2^n}{(n+1)!}$$

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}}{(n+2)!}}{\frac{2^n}{(n+1)!}}\right| \rightarrow \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+2)!} * \frac{(n+1)!}{2^n}\right|$$

After substitution, the fraction can be rewritten so that the numerator is multiplied by the reciprocal of the denominator as shown above.

Step 2: Simplify and solve the limit.

$$\lim_{n\to\infty} \left|\frac{2^n * 2}{(n+2)!} * \frac{(n+1)!}{2^n}\right|$$

A factorial can be expanded by multiplying the initial term with a term that is one less than that of the preceding term. For example:

$$(n+2)! = (n+2)(n+1)(n)(n-1)(n-2)!$$

In the case of the Ratio Test, the factorial should be expanded until the factorials in the numerator and denominator completely canceled out.

Expand the factorial and simplify.

$$\lim_{n\to\infty}\left|\frac{2}{(n+2)(n+1)!} * \frac{(n+1)!}{1}\right| \to \lim_{n\to\infty}\left|\frac{2}{n+2} * 1\right|$$

Evaluate the limit as n approaches infinity

$$r = \lim_{n\to\infty}\left|\frac{2}{\infty + 2}\right| = \frac{2}{\infty} = 0$$

Step 3: Determine the behavior of the series based on the conditions listed in the test.
Through the condition of the Ratio Test, $r < 1$, the original series converges.

Root Test

Similar to the Ratio Test, this test should be used if none of the previous test are applicable. The Root Test differs from the Ratio Test in that it is most effective for series that have an expression to the power of n.

Root Test
Given $\sum_{n=1}^{\infty} a_n$

$$L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}|a_n|^{\frac{1}{n}}$$

Converges: For $L < 1$ Diverges: For $L > 1$ Inconclusive: For $L = 1$

Steps to apply:

Step 1: Apply the formula for the given series.
Step 2: Simplify and solve the limit.
Step 3: Determine the behavior of the series based on the conditions listed in the test.

Example A: Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{2n^n}{3^{n^2+n}}$$

Step 1: Apply the limit for the series.

The limit can be written in two ways:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{2n^n}{3^{n^2+n}}\right|} \leftrightarrow \lim_{n\to\infty} \left|\frac{2n^n}{3^{n^2+n}}\right|^{\frac{1}{n}}$$

Step 2: Simplify and solve the limit.

Raise the numerator and denominator to $\frac{1}{n}$, and simplify

$$\lim_{n\to\infty} \left|\frac{(2n^n)^{\frac{1}{n}}}{\left(3^{n^2} * 3^n\right)^{\frac{1}{n}}}\right| \to \lim_{n\to\infty} \left|\frac{(2n)^{n*\frac{1}{n}}}{(3)^{n^2*\frac{1}{n}} * (3)^{n*\frac{1}{n}}}\right| \to \lim_{n\to\infty} \left|\frac{2n}{3^n * 3}\right|$$

Evaluate the limit as n approached infinity.

$$\lim_{n\to\infty} \left|\frac{2(\infty)}{3^{(\infty)} * 3}\right| = \frac{\infty}{\infty}$$

Apply LHR.

$$\lim_{n\to\infty} \left|\frac{2n}{3^n * 3}\right| \to \lim_{n\to\infty} \left|\frac{2}{\ln(3) * 3^n * 3}\right|$$

Evaluate the limit.

$$\lim_{n\to\infty} \left|\frac{2}{\ln(3) * 3^{(\infty)} * 3}\right| = \frac{2}{\infty} = 0$$

Step 3: Determine the behavior of the series based on the conditions listed in the test. Through the condition of the Root Test, $L < 1$, the original series converges.

Practice:

Determine if the series converges or diverges. If it is a convergent Geometric Series, determine the value.

1.

$$\sum_{n=1}^{\infty} \frac{n^5}{\sqrt[4]{n^3}}$$

2.

$$\sum_{n=1}^{\infty} 8\left(\frac{5}{7}\right)^n$$

3.

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

4.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

5.

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n + 1}$$

6.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2 - 5}}$$

7.

$$\sum_{n=1}^{\infty} \frac{5n^2 - n^3}{3 + 8n^3}$$

8.

$$\sum_{n=1}^{\infty} \left(\frac{10n^5 + 4n^3}{9n^2 - 8n^5}\right)^n$$

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Solutions:

1. Converges. Use formula of P-series.

2. Converges to 28. Use formula of a Geometric Series.

3. Converges via the Integral Test. Hint: Use u-substitution.

4. Converges via the Ratio Test. $\mathrm{r} = 0$

5. Converges via the Direct Comparison Test. Hint: Compare it with a Geometric Series.

6. Diverges via the Direct Comparison Test. Hint: Compare it with a P-series.

7. Diverges via the Divergence Test.

8. Diverges via the Root Test. $\mathrm{L} = \frac{5}{4}$