GERMANNA
ACADEMIC CENTER
FOR EXCELLENCE

Trigonometric Substitution

Integration is a skill that is used frequently in higher level math, physics, and engineering courses. This handout will cover integration using trigonometric substitution, commonly referred to as "trig-sub." For information about other types of integration, please refer to the Academic Center for Excellence's Common Derivatives and Integrals handout.

When to Use Trigonometric Substitution?

Trig-sub is used when the given integral contains a radical that cannot be simplified. The form for such expressions is $\sqrt{\pm a \pm x^2}$. Trig-sub consists of using trigonometric identities to simplify these expressions. The identities that need to be applied in order to solve these integrals are shown in the table below:

Trigonometric Identities	Radical Form
$\cos^2(\theta) = 1 - \sin^2(\theta)$	$\sqrt{a - x^2}$
$\sec^2(\theta) = 1 + \tan^2(\theta)$	$\sqrt{a + x^2}$
$\tan^2(\theta) = \sec^2(\theta) - 1$	$\sqrt{x^2 - a}$

How to Use Trigonometric Substitution?

Trig-sub is done by replacing the expression under the radical with a trigonometric identity that can be simplified. The trigonometric function that is used depends on the form of the radical in the integral. After the integration is performed in the radian domain, it then becomes necessary to use a right triangle to convert the answer back into the x domain.

Step 1: Determine which trigonometric identity to use.

Step 2: Find x^2, x, and dx using the trigonometric identity.

Step 3: Substitute x^2, x, and dx into the original integral.

Step 4: Using the identity from Step 1, replace the expression under the radical.

Step 5: Simplify and solve the integral in the radian domain.

Step 6: Draw a right triangle using x from Step 2 and find the missing side using Pythagorean's Theorem.

Step 7: Using the triangle, convert the solution to the x domain.

Example 1. Solve $\int \frac{x}{\sqrt{9-x^2}} dx$

Step 1: Determine which trigonometric identity to use:

The radical expression is in the form $a - x^2$, and the trigonometric identity that resembles this form is $\cos^2(\theta) = 1 - \sin^2(\theta)$.

Step 2: Find x^2, x, and dx using the trigonometric identity:

Using the identity found in the previous step, x^2 is represented by the trigonometric function, $\sin^2(\theta)$. Note that the constant (a) is 9 in this example. The constant multiplied by $\sin^2(\theta)$ is equal to x^2:

$$x^2 = 9\sin^2(\theta)$$

To determine x, take the square root of both x^2 and $9\sin^2(\theta)$:

$$\sqrt{x^2} = \sqrt{9\sin^2(\theta)}$$

$$x = 3\sin(\theta)$$

Take the derivative of x to find dx:

$$dx = 3\cos(\theta)\, d\theta$$

Step 3: Substitute x^2, x, and dx into the original integral:

$$\int \frac{x}{\sqrt{9 - x^2}} dx \quad \rightarrow \quad \int \frac{\mathbf{3\sin(\theta)}}{\sqrt{9 - \mathbf{9\sin^2(\theta)}}} \mathbf{3\cos(\theta)\, d\theta}$$

Step 4: Using the identity from Step 1, replace the expression under the radical:

$$9 - 9\sin^2(\theta) = 9(1 - \sin^2(\theta)) = 9\cos^2(\theta)$$

$$\int \frac{3\sin(\theta)}{\sqrt{\mathbf{9\cos^2(\theta)}}} 3\cos(\theta)\, d\theta$$

Step 5: Simplify and solve the integral in the radian domain:

$$\int \frac{3\sin(\theta)}{\sqrt{9\cos^2(\theta)}} 3\cos(\theta)\, d\theta = \int \frac{3\sin(\theta)\, 3\cos(\theta)}{3\cos(\theta)} d\theta = \int 3\sin(\theta)\, d\theta$$

$$= -3\cos(\theta) + C$$

Step 6: Draw a right triangle using x from Step 2 and find the missing side using Pythagorean's Theorem:

Recall, from Step 2, $\mathrm{x} = 3\sin\theta$.

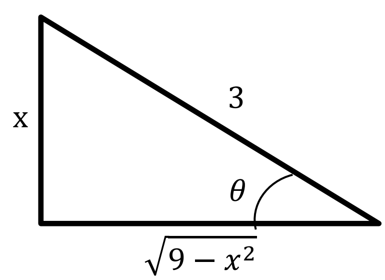
Solving for the trig function gives $\frac{\mathrm{x}}{3} = \sin\theta$.

Remember that $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$. The opposite side of the right triangle is x and the hypotenuse is 3. Use the Pythagorean Theorem to determine the missing side in the triangle:

$$\mathrm{x}^2 + \mathrm{b}^2 = 3^2$$
$$\mathrm{b}^2 = 9 - \mathrm{x}^2$$
$$\mathrm{b} = \sqrt{9 - \mathrm{x}^2}$$

The adjacent side of the triangle is $\sqrt{9 - \mathrm{x}^2}$.

Note that the missing side is equal to the radical in the original integral. This can be used to check that x was found correctly.



Step 7: Using the triangle, convert the solution to the x domain:

The solution to the integral in the theta domain is $-3\cos(\theta) + \mathrm{C}$.

Because $\cos = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\cos(\theta) = \frac{\sqrt{9-\mathrm{x}^2}}{3}$.

Substitute this into the solution in the theta domain:

$$-3\left(\frac{\sqrt{9-\mathrm{x}^2}}{3}\right) + \mathrm{C} = -\sqrt{9 - \mathrm{x}^2} + \mathrm{C}$$

Therefore, $\int \frac{\mathrm{x}}{\sqrt{9-\mathrm{x}^2}}\,\mathrm{dx} = -\sqrt{9 - \mathrm{x}^2} + \mathrm{C}$

Example 2. Solve $\int_1^2 \frac{\sqrt{x^2+25}}{x^4}dx$

Step 1: Determine which trigonometric identity to use:

The radical expression is in the form $x^2 + a$, and the trigonometric identity that resembles this form is $\sec^2(\theta) = \tan^2(\theta) + 1$.

Step 2: Find x^2, x, and dx using the trigonometric identity:

Using the identity found in the previous step, x^2 is represented by the trigonometric function, $\tan^2(\theta)$. Note that the constant (a) is 25 in this example. The constant multiplied by $\tan^2(\theta)$ represents x^2:

$$x^2 = 25\tan^2(\theta)$$

To determine x, square root both x^2 and $25\tan^2(\theta)$:

$$\sqrt{x^2} = \sqrt{25\tan^2(\theta)}$$
$$x = 5\tan(\theta)$$

Take the derivative of x to find dx:

$$dx = 5\sec^2(\theta)\,d\theta$$

Step 3: Substitute x^2, x, and dx into the original integral:

$$\int_1^2 \frac{\sqrt{x^2+25}}{x^4}dx \to \int_{x=1}^{x=2} \frac{\sqrt{25\tan^2(\theta)+25}}{625\tan^4(\theta)}5\sec^2(\theta)d\theta$$

Step 4: Using the identity from Step 1, replace the expression under the radical:

$$\int_{x=1}^{x=2} \frac{\sqrt{25\tan^2(\theta)+25}}{625\tan^4(\theta)}5\sec^2(\theta)d\theta = \int_{x=1}^{x=2} \frac{\sqrt{25(\sec^2(\theta))}}{625\tan^4(\theta)}5\sec^2(\theta)d\theta$$

Step 5: Simplify and solve the integral in the radian domain:

$$\int_{x=1}^{x=2} \frac{\sqrt{25(\sec^2(\theta))}}{625\tan^4(\theta)} 5\sec^2(\theta)d\theta = \int_{x=1}^{x=2} \frac{5\sec(\theta)}{625\tan^4(\theta)} 5\sec^2(\theta)d\theta$$

$$= \int_{x=1}^{x=2} \frac{25\sec^3(\theta)}{625\tan^4(\theta)} d\theta = \int_{x=1}^{x=2} \frac{\sec^3(\theta)}{25\tan^4(\theta)} d\theta$$

$$= \frac{1}{25}\int_{x=1}^{x=2} \frac{\frac{1}{\cos^3(\theta)}}{\frac{\sin^4(\theta)}{\cos^4(\theta)}} d\theta = \frac{1}{25}\int_{x=1}^{x=2} \frac{\cos(\theta)}{\sin^4(\theta)} d\theta$$

$$= \frac{1}{25}\int_{x=1}^{x=2} \cot(\theta)\csc^3(\theta)d\theta$$

$$u = \csc(\theta)$$

$$du = -\csc(\theta)\cot(\theta)\,d\theta$$

$$\frac{-1}{25}\int_{x=1}^{x=2} u^2du = \frac{-1}{25}\left(\frac{1}{3}u^3\right)\Big|\begin{matrix} x=2 \\ x=1\end{matrix} = \frac{-1}{75}(\csc(\theta)^3)\Big|\begin{matrix} x=2 \\ x=1\end{matrix}$$

Step 6: Draw a right triangle using x from Step 2 and find the missing side using Pythagorean's Theorem:

Recall from Step 2, $x = 5\tan(\theta)$.

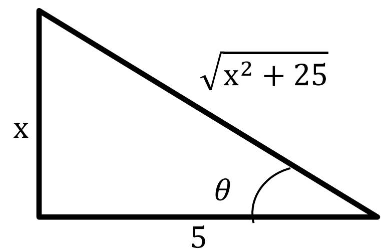
Solving for the trig function gives $\tan(\theta) = \frac{x}{5}$.

Remember that $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$. The opposite side of the right triangle is x and the adjacent is 5. The missing side in this triangle is the hypotenuse:

$$x^2 + 5^2 = c^2$$

$$c = \sqrt{x^2 + 25}$$

The hypotenuse of the triangle is $\sqrt{x^2 + 25}$.



Step 7: Using the triangle, convert the solution to the x domain:

The solution to the integral in the theta domain is $\frac{-1}{75}(\csc(\theta)^3)\Big|_{x=1}^{x=2}$.

Because $\csc = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}$, $\csc(\theta) = \frac{\sqrt{x^2+25}}{x}$.

Substitute this into the solution in the theta domain:

$$\frac{-1}{75}(\csc(\theta)^3)\Big|_{x=1}^{x=2} = \frac{-1}{75}\left(\frac{\sqrt{x^2+25}}{x}\right)^3 \Big|_1^2$$

Because the answer is now in the x domain, the bounds of integration can be used to find the solution to the definite integral.

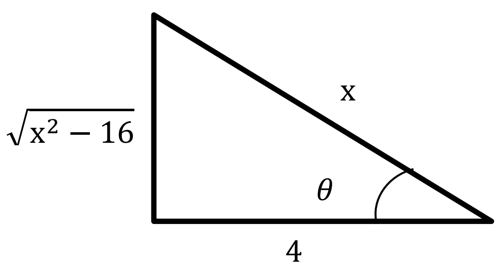
$$\boxed{\frac{-\left(\left(\frac{\sqrt{29}}{2}\right)^3 - \left(\sqrt{26}\right)^3\right)}{75} = \frac{26^{3/2} - \frac{29^{3/2}}{8}}{75} \approx 1.51}$$

Practice Problems:

1. Solve $\int \frac{1}{x^2\sqrt{x^2-16}}\,dx$
2. Solve $\int_0^1 \frac{x^2}{\sqrt{9-x^2}}\,dx$

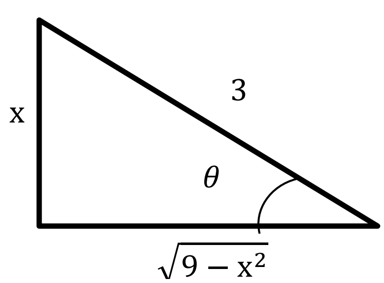
Solutions:

1. Triangle:



Solution: $\frac{\sqrt{x^2-16}}{16x} + C$

2. Triangle:



Solution: $\frac{9\sin^{-1}\left(\frac{x}{3}\right)}{2} - \frac{x\sqrt{9-x^2}}{2}\Big|_0^1 = 0.1151$