Radicals and Fractional Exponents

- **Radicals and Roots**

  In math, many problems will involve what is called the radical symbol, $\sqrt{\ }$

  $\sqrt[n]{X}$ is pronounced the nth root of X, where n is 2 or greater, and X is a positive number. What it is asking you is what number multiplied by its self "n" number of times will equal X? Ex.

Square Roots	Cube Roots	4^{th} Roots
$\sqrt[2]{4}$ =2 because 2x2=4	$\sqrt[3]{8}$ = 2, because 2x2x2= 8	$\sqrt[4]{16}$ = 2, because 2x2x2x2=16
$\sqrt[2]{36}$ = 6, because 6x6=36	$\sqrt[3]{27}$ = 3, because 3x3x3=27	$\sqrt[4]{81}$ = 3, because 3x3x3x3=81

- **Perfect Roots**

  Perfect roots are roots that can be perfectly broken down like in the examples above.
  Here is a list of the most common perfect roots. These should be memorized!

Perfect Square Roots		Perfect Cube, Fourth, and Fifth Roots	
$\sqrt{1} = 1$	$\sqrt{64} = 8$	$\sqrt[3]{1} = 1$	$\sqrt[4]{81} = 3$
$\sqrt{4} = 2$	$\sqrt{81} = 9$	$\sqrt[3]{8} = 2$	$\sqrt[4]{256} = 4$
$\sqrt{9} = 3$	$\sqrt{100} = 10$	$\sqrt[3]{27} = 3$	$\sqrt[4]{625} = 5$
$\sqrt{16} = 4$	$\sqrt{121} = 11$	$\sqrt[3]{64} = 4$	$\sqrt[5]{1} = 1$
$\sqrt{25} = 5$	$\sqrt{144} = 12$	$\sqrt[3]{125} = 5$	$\sqrt[5]{32} = 2$
$\sqrt{36} = 6$	$\sqrt{169} = 13$	$\sqrt[4]{1} = 1$	$\sqrt[5]{243} = 3$
$\sqrt{49} = 7$		$\sqrt[4]{16} = 2$	

- **Solving Imperfect Radical Expressions**

  Imperfect radical expressions are numbers that do not have perfect roots. For example $\sqrt[2]{5}$, there is no number that when multiplied by itself will give you 5, except a decimal. However, we still have to simplify them as much as we can. The easiest way to do it is to break the number down into a product of its primes by using a factor tree. Once that is done, every number that repeats itself n number of times can be pulled out of the radical, everything else remains inside.

Ex. $\sqrt[2]{12}$ = ?	
Step 1. Break down into products of primes	**Step 2. Look number repeating n times**
12 /\\ 6 x 2 /\\ \\ 3 x 2 x 2	N = 2 so look for number that repeats twice. 3 x 2 x 2 → 3 x 2 x 2 **Step 3. Pull out of Radical** 2 goes in front of radical, and 3 is left underneath. $2\sqrt[2]{3}$

If more than one number can be pulled out from the radical, then you multiply them on the outside.

Ex. $\sqrt[3]{128}$ =?	
128 / \ 4 x 32 /\ /\ 2x2x4x8 / / /\/\ 2x2x2x2x4 / / / / / / \ 2x2x2x2x2x2x2	N=3, so look for number repeating 3 times. 2x2x2x2x2x2x2=128 Pull each group out and put in front of radical sign and multiply. 2x2$\sqrt[3]{2} \rightarrow 4\sqrt[3]{2}$

Another way of solving imperfect radical expressions is to break the number down into a product of perfect squares (this is why it is important to have them memorized!). Then you can solve each perfect square individually, for Ex. $\sqrt[2]{288}$ =$\sqrt[2]{36x4x2}$ =$\sqrt[2]{36}$ x $\sqrt[2]{4}$ x $\sqrt[2]{2}$ = 6x2x$\sqrt[2]{2}$=12$\sqrt[2]{2}$

Ex. $\sqrt[2]{72}$ =?	
Step 1. Break down into a product of perfect squares	**Step 2. Simplify perfect squares individually, and leave what can't be broken down further under the radical.**
72 / \ 9 8 / /\ 9 4 2	$\sqrt[2]{72}$=$\sqrt[2]{9}$ x $\sqrt[2]{4}$ x $\sqrt[2]{2}$ → $\sqrt[2]{72}$=3 x 2 x $\sqrt[2]{2}$ **Step 3. Multiply numbers on the outside of radical.** $\sqrt[2]{72}$=3 x 2 x $\sqrt[2]{2}$ → 6$\sqrt[2]{2}$

Ex. $\sqrt[3]{432}$ =?	
432 / \ 2 216 / / \ 2 8 27	1. $\sqrt[3]{432}$ = $\sqrt[3]{8}$ x $\sqrt[3]{27}$ x $\sqrt[3]{2}$ 2. $\sqrt[3]{432}$ = 2 x 3 x $\sqrt[3]{2}$ 3. . $\sqrt[3]{432}$ = 6$\sqrt[3]{2}$

- **Radical expressions with variables**

  Some radical expressions will also include variables, ex. $\sqrt[2]{216a^4b^3}$. To simplify, treat the numbers as always. The variables can be simplified by dividing "n" into the exponent of the variable. However many times it is evenly divisible is how many you can take out; leave the remainder under the radical. For example, $\sqrt[3]{a^7}$. N=3, and 3 goes into 7 twice with one left over, so then I take two a's out and leave one under the radical, $a^2\sqrt[3]{a}$.

Ex. $\sqrt[3]{32a^3b^8}$ = ?

Step 1. Break down number	Step 2. Break down the "a"	Step 3. Break down the "b"
32⟶8x4⟶2x2x2x2x2, so one 2 on the outside, two inside	3 goes into 3 once with zero left over. So one <u>a</u> on the outside, none inside.	3 goes into 8 twice with two left over. So two <u>b's</u> on the outside, and two inside
Final answer $2ab^2\sqrt[3]{4b^2}$		

- **Adding and Subtracting Radical Expressions**
  When adding or subtracting radicals you treat them the same as you would a variable, you can only put like terms together. Both the index, i.e. the n value, and what is under the radical must be identical in order to add or subtract. Just like 3a + 2a = 5a, $2\sqrt[3]{6} + 4\sqrt[3]{6} = 6\sqrt[3]{6}$.

- **Multiplying Radical Expressions**
  When multiplying radical expression you simply need to follow this rule, $\sqrt[n]{a}$ x $\sqrt[n]{b}$ = $\sqrt[n]{a \text{ x } b}$. If there are coefficients, you simply multiply them normally. The final step is to simplify if possible. Also, remember that in order to multiply, the index must be the same, you cannot multiply a square root with a cube root.

Ex. $2ab^2\sqrt[3]{9c^2}$ x $4a^3\sqrt[3]{18b^4}$ = ?		
Step 1. Multiply coefficients	**Step 2. Multiply under radical**	**Step 3. Can you simplify?**
$2ab^2$ x $4a^3$ = $8a^4b^2$	$\sqrt[3]{9c^2}$ x $\sqrt[3]{18b^4}$ = $\sqrt[3]{162b^4c^2}$	$8a^4b^2\sqrt[3]{162b^4c^2}$ yes
Step 4. Simplify		
$8a^4b^2\sqrt[3]{162b^4c^2}$ ⟶ $8a^4b^2\sqrt[3]{6\text{x}27b^4c^2}$ ⟶ $8\text{x}3a^4b^2\sqrt[3]{6b^4c^2}$ ⟶ $24a^4b^3\sqrt[3]{6bc^2}$		

- **Dividing Radical Expressions**
  When dividing radical expressions you need to follow this rule, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$. If there are coefficients, you simply divide them normally. Then simplify what is under the radical as much as possible, and then simplify the radical itself if possible. Remember, in order to divide the degree must be the same for both radical expressions.

Ex. $\sqrt[3]{162a^7b^5} \div \sqrt[3]{3a^3b^4}$ = ?		
Step 1. Rewrite as 1 Radical	**Step 2. Simplify under Radical**	**Step 3. Simplify Radical**

$\sqrt[3]{\dfrac{162a^7b^5}{3a^3b^4}}$	$\sqrt[3]{54a^4b}$	$\sqrt[3]{54a^4b} \rightarrow \sqrt[3]{27 \times 2a^4b}$ $\rightarrow$ 3a$\sqrt[3]{2ab}$

- **Exponents**

  Exponents are very much like the reverse of roots. Rather than what number multiplied by itself n number of times equals X as with the radical $\sqrt[n]{X}$, X^n is asking X multipled by itself n number of times equals what? For example 3^4 = 81 because 3x3x3x3=81. Notice that $\sqrt[4]{81}$ = 3. Here are some rules and properties for working with exponents.

Adding and Subtracting	**Multiplying**	**Dividing**
Must be same degree, only add/subtract the coefficients. Ex. $2x^3 + 3x^3 = 5x^3$	Add the exponents, $a^n x\ a^m = a^{(n+m)}$ Ex.$6a^3$ x $3a^2$ = $18a^5$	Subtract exponents, $a^n /\ a^m = a^{(n-m)}$ Ex. $6a^4/3a^2$ = $2a^2$

Power to power	**Negative Exponents**
Multiply the exponents for the variable, apply exponent to coefficient. $(a^n)^m$ = a^{nxm} Ex. . $(3a^3)^4$ = 3^4a^{12} = $81a^{12}$	Move from numerator to denominator or vice versa to make exponent positive. $x^{-n} = 1/\ x^n$ Ex. $(5/3)^{-3}$ = $5^{-3}/3^{-3}$ = $3^3/5^3$ = 27/125

- **Fractional Exponents**

  Fractional Exponents must be simplified a different way than normal exponents. For example, $4^{1/2}$. You cannot multiply 4 by its self ½ times. Since Radicals and exponents are reverses of each other, we can switch from exponential form to radical form to simplify. In order to do that, simply follow this formula: $x^{n/m} = \sqrt[m]{x^n}$.

Ex. $16^{1/2} = \sqrt[2]{16}$ = 4	Ex. $4^{3/4}$ = $\sqrt[4]{4^3}$ = $\sqrt[4]{64}$ =$\sqrt[4]{16 \times 4}$ = 2$\sqrt[4]{4}$

- Practice Problems (Simplify)

1. $\sqrt[2]{32x^4y^7}$
2. $\sqrt[3]{8x^3y^6}$
3. $\sqrt[4]{81a^8b^{12}}$
4. $\sqrt[5]{64a^8b^{12}}$
5. $4x\sqrt[2]{12x^2y} + \sqrt[2]{3x^4y} - x^2\sqrt[2]{27y}$
6. $\sqrt[3]{54x^7y^3} - x\sqrt[3]{128x^4y^3} - x^2\sqrt[3]{2xy^3}$
7. $\sqrt[3]{16x^4y} \times \sqrt[3]{4xy^5}$
8. $\left(2x^{1/3}y^{-2/3}\right)^6 / (x^{-4}y^8)^{\frac{1}{4}}$
9. $\sqrt[2]{65ab^4} / \sqrt[2]{5ab}$
10. $\sqrt[4]{12ab^3} \times \sqrt[4]{4a^5b^2}$
11. $8^{1/3}$
12. $(25/49)^{-3/2}$
13. $a^{1/3} \times a^{5/3}$
14. $\left(x^{-\frac{2}{3}}\right)^6$
15. $(a^3b^9)^{2/3}$

- Solutions

1. $4x^2y^3\sqrt[2]{2y}$
2. $2xy^2$
3. $3a^2b^3$
4. $2ab^2\sqrt[2]{2a^3b^2}$
5. $6x^2\sqrt[2]{3y}$
6. $-2x^2y\sqrt[3]{2x}$
7. $4xy^2\sqrt[3]{x^2}$
8. $64x^3/y^6$
9. $b\sqrt[2]{13b}$
10. $2ab\sqrt[4]{3a^2b}$
11. 2
12. $343/125$
13. a^2
14. $1/x^4$
15. a^2b^6