Partial Derivatives

Partial derivatives are a way to derive functions that have more than one independent variable. They have a wide range of uses, including topics in Physics, Calculus, Economics, and Computer Science. This handout will focus on the fundamental techniques for solving differential equations using partial derivatives.

Notation

Partial derivatives use a notation that is intentionally similar to that of regular derivatives. Their overall format is the same, but the shorthand symbols, such as "dx," are replaced by the stylized version of the letter d: ∂. Partial derivatives should be labeled as $\frac{\partial}{\partial \mathrm{x}}$ or $\frac{\partial}{\partial \mathrm{y}}$, depending on the variable being derived. The two derivatives should be read as "The partial derivative, with respect to x," and "The partial derivative, with respect to y."

Process for Solving

Partial derivatives follow the same procedures and rules of differentiation as normal derivatives with one exception. When taking a partial derivative with respect to a particular variable, treat all other variables as though they are constants such as when deriving with respect to x, treat y as a constant, and do not derive the y. When deriving with respect to y, do not derive x. This method may at first seem similar to implicit differentiation; however, there is a crucial difference between the two methods. Implicit differentiation can be used when y depends on the independent x variable. When each variable is independent of the other, the partial derivative method should be used.

This handout will only cover using the Constant, Product, and Power Rules of differentiation for solving a partial derivative. However, a complete list of the rules can be found in any Calculus textbook or in our Common Derivatives and Integrals handout, which is also located on the

Academic Center for Excellence website at www.germanna.edu/academic-center-for-excellence/helpful-handouts.

The mathematical definitions of these three rules of differentiation are as follows:

Constant Rule: $(a)' = 0$

Product Rule: $\left(f(x) \cdot g(x)\right)' = f(x) \cdot g(x)' + f(x)' \cdot g(x)$

Power Rule: $(ax^b)' = abx^{b-1}$

Example Problem

Take the partial derivatives of the following function with respect to x and then y:

$$f(x, y) = x^2y + 2x^3$$

To take the partial derivative of this function with respect to x, apply the Product, Power, and Constant Rules to the x^2y term, and apply the Power Rule to the $2x^3$ term.

To apply the Product Rule to the x^2y term, let $f(x)$ represent x^2 and $g(x)$ represent y. First, derive $f(x)$ by using the Power Rule to yield a derivative of $2x$. Secondly, derive $g(x)$. Because $g(x)$ represents y, and the partial derivative being taken is with respect to x, $g(x)$ will be considered a constant per the Constant Rule and yield a derivative of 0. Combining the $f(x)$ and $g(x)$ pieces of the Product Rule will yield the following:

$$\frac{\partial f}{\partial x}(x^2y) = 2xy + x^2 \cdot 0 = 2xy$$

Applying the Power Rule to the equation's second term will yield $6x^2$. After combining both parts, the final result of the partial derivative with respect to x is:

$$\frac{\partial f}{\partial x} = 2xy + 6x^2$$

When taking the partial derivative with respect to y, the Power, Product, and Constant Rules are used to find the derivative of the first term, and the Constant Rule alone is applied to find the derivative of the second term.

For the first term, the x^2 and y still correspond to $f(x)$ and $g(x)$ respectively; however, their derivatives have changed. The Constant Rule must be used to derive x^2 because variables other than y are now treated as constants. Use the Power Rule to take the derivative of y, which will result in a value of 1. Finally, the two halves of the Product Rule can be reassembled.

$$\frac{\partial f}{\partial y}(x^2y) = 0 \cdot y \; + \; x^2 \cdot 1 = x^2$$

When deriving the second term, only the Constant Rule is needed since $2x^3$ does not contain any y variables. The derivative will, therefore, be 0. Combining the terms together will give the final result for the partial derivative with respect to y.

$$\frac{\partial f}{\partial y} = x^2 + 0 = x^2$$

This handout specifically illustrates how to use the Constant, Power, and Product Rules when taking a partial derivative, but the same process can also be applied to any of the other rules of differentiation. For any problem, identify the variable that needs to be derived, then treat all other variables as constants and apply any derivative rules that are necessary to solve the equation.

Practice:

Find the partial derivatives with respect to each variable present:

1. $f(x,y) = 3x^2y$

2. $S(l,w,\mathrm{V}) = 2lw + \dfrac{2V}{l} + \dfrac{2V}{w}$

3. $l(x,T) = 2e^{2x}T^{-4}$

4. $p(T,v,h) = T^2hv^{-1} + 7hv - T$

5. $f(x,y,z) = \sin(3x)e^{2y}z^4$

Answers to the Above Problems

1. $\frac{\partial f}{\partial x} = 6\text{x}y, \quad \frac{\partial f}{\partial y} = 3x^2$

2. $\frac{\partial S}{\partial l} = 2w - \frac{2V}{l^2}, \quad \frac{\partial S}{\partial w} = 2l - \frac{2V}{w^2}, \quad \frac{\partial S}{\partial V} = \frac{2}{l} + \frac{2}{w}$

3. $\frac{\partial l}{\partial x} = 4e^{2x}T^{-4}, \quad \frac{\partial l}{\partial T} = -8e^{2x}T^{-5}$

4. $\frac{\partial p}{\partial T} = 2Thv^{-1} - 1, \quad \frac{\partial p}{\partial v} = -T^2hv^{-2} + 7h, \quad \frac{\partial p}{\partial h} = T^2v^{-1} + 7v$

5. $\frac{\partial f}{\partial x} = 3\cos(3x)e^{2y}z^4, \quad \frac{\partial f}{\partial y} = 2\sin(3x)e^{2y}z^4, \quad \frac{\partial f}{\partial z} = 4\sin(3x)e^{2y}z^3$