Laplace Transforms

A Laplace transform converts between the frequency (s) domain and time (t) domain using integration and is commonly used to solve differential equations. There are two ways to find the Laplace transform: integration and using common transforms from a table. This handout will cover both Laplace transform methods, inverse Laplace transforms, and using transforms to solve initial value differential equation problems (IVPs).

You can navigate to specific sections of this handout by clicking the links below.

Laplace Transform Methods

The *integration method* can be used for any function as long as the integral converges or the limit exists. The Laplace transform integral is:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Example: Find the Laplace transform of f(t) = 1 using the integration method.

$$\mathcal{L}\{1\} = \int_0^b e^{-st}(1) dt$$

$$\mathcal{L}\{1\} = \lim_{b \to \infty} \frac{-e^{-st}}{s} \Big|_0^b$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

Another way to solve a Laplace transform is to use *common transforms* listed within a Laplace Transform table. These are transforms that have already been calculated using the integration method. Common Laplace transforms can often be obtained from textbooks or from the Internet. The transforms below will be referenced throughout this handout.

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1. $\mathscr{L}\{t\} = \frac{1}{s^2}$	5. $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$
2. $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$, n is a positive integer	6. $\mathscr{L}\{te^{at}\} = \frac{1}{(s-a)^2}$
3. $\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$	7. $\mathscr{L}\{e^{at}\sin(kt)\} = \frac{k}{(s-a)^2+k^2}$
4. $\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$	8. $\mathscr{L}\{e^{at}\cos(kt)\} = \frac{s-a}{(s-a)^2+k^2}$

Example: Find the Laplace transform of the function, $f(t) = \cos(2t)$, using common Laplace transforms.

$$\mathscr{L}\{f(t) = \cos(2t)\}$$

$$F(s) = \frac{s}{s^2 + 2^2}$$ Use transform 4 from the table

$$\boxed{F(s) = \frac{s}{s^2 + 4}}$$

An important application of the common transform method is the Laplace transform of a derivative:

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

While reviewing the transform above, note that the first s exponent matches the order of the differential equation. The first term is multiplied by the Laplace transform of the function, and the second term is multiplied by the function at t=0. Each subsequent s term is multiplied by the derivative in increasing order. Continue to reduce the exponent of s by one until the exponent is 0. This will be the last term of the transform. In order to use the Laplace Transform for an IVP problem, the initial of values of $f(0)$ must be provided. This topic will be discussed later within this handout on page 5.

Example: Find the Laplace transform of y'''

$$\mathscr{L}\{y'''\} = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$

Laplace transforms possess a linearity, meaning a function can be broken into pieces in order to apply the common transforms. Here is another example:

Example: Find the Laplace transform of the function, $y(t) = 1 + 2t - 3t^3$

$\mathcal{L}\{y(t) = 1 + 2t - 3t^3\}$

Break it down into separate functions and add them together

$\mathcal{L}\{y(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{2t\} - \mathcal{L}\{3t^3\}$

Take the Laplace of each function using the table

$Y(s) = \dfrac{1}{s} + 2\dfrac{1}{s^2} - 3\dfrac{3!}{s^{3+1}}$

Simplify

$\boxed{Y(s) = \dfrac{1}{s} + \dfrac{2}{s^2} - \dfrac{18}{s^4}}$

Inverse Laplace Transforms

Inverse Laplace transforms are the opposite of Laplace transforms. Inverse Laplace transforms convert from the s domain to the t domain. In most cases, the function does not look exactly like one of the transforms from the table, so it will have to be manipulated first.

Example: Find the inverse Laplace of the function, $G(s) = \dfrac{3s+2}{s^2+9}$

$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\dfrac{3s+2}{s^2+9}\right\}$

Separate into components

$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\dfrac{3s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\dfrac{2}{s^2+9}\right\}$

$\mathcal{L}^{-1}\{G(s)\} = 3\,\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}\right\} + 2\,\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+9}\right\}$

Move the coefficient outside of the inverse Laplace

$\mathcal{L}^{-1}\{G(s)\} = 3\,\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}\right\} + \dfrac{2}{3}\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+9}\right\}$

Match each component to a Laplace transform Multiply by a number if necessary

$\boxed{g(t) = 3\cos(3t) + \dfrac{2}{3}\sin(3t)}$

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Partial fractions are useful in finding inverse Laplace transforms. Partial fractions separate a rational expression into components. Once in components, the Laplace transform can be calculated using the table of common transforms. Use the table below to set up the partial fractions.

Factor in Denominator	Partial Fraction Term
$(s - a)$	$\dfrac{A}{s - a}$
$(s - a)^k$	$\dfrac{A}{(s - a)} + \dfrac{B}{(s - a)^2} + \cdots + \dfrac{C}{(s - a)^k}$
$(s^2 + as + b)$	$\dfrac{As + B}{(s^2 + as + b)}$

Example: Find the Inverse Laplace transform of $\dfrac{2s^2+3s-8}{(s+2)(s-1)(s+6)}$

$$\frac{2s^2 + 3s - 8}{(s + 2)(s - 1)(s + 6)} = \frac{A}{(s + 2)} + \frac{B}{(s - 1)} + \frac{C}{(s + 6)}$$
Set up partial fractions

$$2s^2 + 3s - 8 = A(s - 1)(s + 6) + B(s + 2)(s + 6) + C(s + 2)(s - 1)$$
Multiply each fraction by the left denominator

$$2s^2 + 3s - 8 = A(s^2 + 5s - 6) + B(s^2 + 8s + 12) + C(s^2 + s - 2)$$

$$2s^2 + 3s - 8 = As^2 + 5As - 6A + Bs^2 + 8Bs + 12B + Cs^2 + Cs - 2C$$
Simplify

$$\begin{cases} s^2 \colon 2 = A + B + C \\ s^1 \colon 3 = 5A + 8B + C \\ s^0 \colon -8 = -6A + 12B - 2C \end{cases}$$
Pair coefficients based on s^n

$$A = \frac{1}{2}, B = -\frac{1}{7}, C = \frac{23}{14}$$
Solve the system of equations for A, B, and C

$$\mathscr{L}^{-1}\left\{\frac{s^2+3s-8}{(s+2)(s-1)(s+6)}\right\} = \mathscr{L}^{-1}\left\{\frac{1/2}{(s+2)} - \frac{1/7}{(s-1)} + \frac{23/14}{(s+6)}\right\}$$
Replace A, B, and C with the calculated values

4

$$\mathscr{L}^{-1}\left\{\frac{s^2+3s-8}{(s+2)(s-1)(s+6)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\frac{1}{(s+2)} - \frac{1}{7}\frac{1}{(s-1)} + \frac{23}{14}\frac{1}{(s+6)}\right\}$$

Match each component to a Laplace transform

$$\boxed{\mathscr{L}^{-1}\left\{\frac{s^2+3s-8}{(s+2)(s-1)(s+6)}\right\} = \frac{1}{2}e^{-2t} - \frac{1}{7}e^{1t} + \frac{23}{14}e^{-6t}}$$

Using Laplace Transforms to Solve IVPs

Laplace transforms can be used to solve IVPs in ordinary differential equations. To solve an IVP using Laplace transforms, use the following five steps. As an example, the following IVP will be used:

$$y'' - 6y' + 8y = e^{-4t} \quad \text{initial conditions: } y(0) = 1 \text{ } and \text{ } y'(0) = 3$$

Step 1: Take the Laplace transform of the differential equation.

$$\mathscr{L}\{y'' - 6y' + 8y = e^{-4t}\}$$

$$[s^2Y(s) - sy(0) - y'(0)] - 6[sY(s) - y(0)] + 8[Y(s)] = \frac{1}{s+4}$$

Step 2: Plug in the initial conditions.

$$[s^2Y(s) - s - 3] - 6[sY(s) - 1] + 8[Y(s)] = \frac{1}{s+4}$$

Step 3: Solve for Y(s).

$$Y(s) = \frac{s^2 + s - 11}{(s-2)(s-4)(s+4)}$$

Step 4: Use partial fractions to separate the radical into components.

$$\frac{s^2 + s - 11}{(s-2)(s-4)(s+4)} = \frac{A}{(s-2)} + \frac{B}{(s-4)} + \frac{C}{(s+4)}$$

$$s^2 + s - 11 = As^2 - 16A + Bs^2 + 2Bs - 8B + Cs^2 - 6Cs + 8C$$

5

$$\begin{cases} s^2: 1 = A + B + C \\ s^1: 1 = 0A + 2B - 6C \\ s^0: -11 = -16A - 8B + 8C \end{cases}$$

$$A = \frac{5}{12}, B = \frac{9}{16}, C = \frac{1}{48}$$

$$Y(s) = \frac{5/12}{s-2} + \frac{9/16}{s-4} + \frac{1/48}{s+4}$$

Step 5: Take the inverse Laplace transform to solve for y(t).

$$\mathscr{L}^{-1}\{Y(s)\} = \frac{5}{12}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{9}{16}\mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\} + \frac{1}{48}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\boxed{y(t) = \frac{5}{12}e^{2t} + \frac{9}{16}e^{4t} + \frac{1}{48}e^{-4t}}$$

Practice Problems

Find the Laplace transform of the following (use either method):

1. $y(t) = te^{3t} + t^2 - t$

2. $y(t) = \sin(6t) + e^{-2t}\sin(4t)$

3. $y(t) = e^{3t} - e^{7t}\cos(9t)$

Find the inverse Laplace transform of the following:

4. $Y(s) = \frac{1}{s^4} - \frac{6}{s-2} + \frac{3}{s^2}$

5. $Y(s) = \frac{2s-6}{(s-3)^2+16} + \frac{3}{(s-5)^2}$

6. $Y(s) = \frac{3s^2+2s-9}{(s^2-16)(s-1)}$

6

Solve the following IVPs using Laplace transforms:

7. $y'' + 9y' + 14y = e^{-12t}$ I.C. $y(0) = 5 \ and \ y'(0) = 10$

8. $y'' + 5y' - 24y = e^{7t}$ I.C. $y(0) = 2 \ and \ y'(0) = 4$

9. $y'' - 5y' + 4y = e^{-5t}$ I.C. $y(0) = 1 \ and \ y'(0) = 0$

Answers

1. $Y(s) = \frac{1}{(s-3)^2} + \frac{2}{s^3} - \frac{1}{s^2}$, use transform 6, 2 and 1

2. $Y(s) = \frac{6}{s^2+36} + \frac{4}{(s+2)^2+16}$, use transform 3 and 7

3. $Y(s) = \frac{1}{s-3} - \frac{s-7}{(s-7)^2+81}$, use transform 5 and 8

4. $y(t) = \frac{1}{6}t^3 - 6e^{2t} + 3t$, use transform 2, 5, and 1

5. $y(t) = 2e^{3t}\cos(4t) + 3te^{5t}$, use transform 8 and 6

6. $y(t) = \frac{31}{40}e^{-4t} + \frac{47}{24}e^{4t} + \frac{4}{15}e^t$, use transform 5

7. $y(t) = 61.82e^{-2t} - 109.64e^{-7t} + 52.89e^{-12t}$

8. $y(t) = \frac{31}{165}e^{-8t} + \frac{79}{44}e^{3t} + \frac{1}{60}e^{7t}$

9. $y(t) = \frac{23}{18}e^t - \frac{8}{27}e^{4t} + \frac{1}{54}e^{-5t}$

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